ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 1, Number 2, Spring 1971

DIVISIBLE QUOTIENT GROUPS OF REDUCED ABELIAN GROUPS

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The groups under discussion here are all Abelian, so *group* means *Abelian group*. A group G is *divisible* if $nG = \{ng \mid g \in G\} = G$ for all nonzero integers n. Typical examples are the additive group Q of rational numbers, and its homomorphic images. In fact, every divisible group is a direct sum of such groups. If D is a divisible subgroup of G, then D is a summand of G. Every group G has a unique largest divisible subgroup D, and if $G = D \oplus H$, then H has no nonzero divisible subgroups. Such an H is called *reduced.* Reduced groups G can have divisible quotient groups G/A. In fact, free groups are reduced and certainly have divisible quotient groups. However, if G is reduced and G/A is divisible, then A cannot be too small compared to G. For example, suppose G is a reduced p-group and B is a *basic* subgroup of G. That is, B is a subgroup of G such that G/B is divisible, B is a direct sum of cyclic groups, and $B \cap nG = nB$ for all integers n. Then Fuchs shows [1, Theorem 30.1] that $|B|^{\aleph_0} \geqq |G|$. Fuchs proves this using his [2] quasibases of such G. He then uses $|B|^{\aleph_0} \geqq |G|$ to show that for reduced p-groups G, $|G/G^1|^{\aleph_0} \geqq |G|$, where $G^1 = \bigcap_{n=1}^{\infty} nG$ is the subgroup of elements of infinite height in G. The facts are important in the theory of p-groups. (They are crucial in establishing necessary and sufficient conditions for a well-ordered sequence of p-groups with no elements of infinite height to be the Ulm sequence of a reduced p-group. See [1, Chapter VI], for example.) Now these inequalities hold in general. That is, if G is any reduced group and G/A is divisible, then $|A|^{\aleph_0} \geqq |G|$, and $|G/G^1|^{\aleph_0} \geqq |G|$. The group G does not have to be a p-group and A does not have to be a basic subgroup of G. The second inequality is actually a consequence of the first, as we shall see. The inequality $|A|^{\aleph_0} \geqq |G|$ has a short homological proof as follows. The exact sequence $0 \to Z \to Q \to Q/Z \to 0$ yields the exact sequence $\mathrm{Hom}(Q, G) = 0 \to \mathrm{Hom}(Z, G) \approx G \to \mathrm{Ext}(Q/Z, G)$ so that $\mathrm{Ext}(Q/Z, G)$ contains a copy of G. The sequence $0 \to A \to G \to G/A \to 0$ yields the epimorphism $\mathrm{Ext}(Q/Z, A) \to \mathrm{Ext}(Q/Z, G) \to 0$; $\mathrm{Ext}(Q/Z, G/A) = 0$ since every extension of a divisible group splits. Thinking of $\mathrm{Ext}(Q/Z, A)$ as the group of factor systems (which are some of the

Received by the editors January 26, 1970.
AMS 1970 *subject classifications.* Primary 20K25; Secondary 20K10, 20D15.