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NUMERICAL SOLUTION VIA LAPLACE TRANSFORMS OF A FRACTIONAL ORDER EVOLUTION EQUATION

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ABSTRACT. We consider the discretization in time of a fractional order diffusion equation. The approximation is based on a further development of the approach of using Laplace transformation to represent the solution as a contour integral, evaluated to high accuracy by quadrature. This technique reduces the problem to a finite set of elliptic equations with complex coefficients, which may be solved in parallel. Three different methods, using $2N+1$ quadrature points, are discussed. The first has an error of order $O\left(e^{-cN}\right)$ away from $t=0$, whereas the second and third methods are uniformly accurate of order $O\left(e^{-c\sqrt{N}}\right)$. Unlike the first and second methods, the third method does not use the Laplace transform of the forcing term. The basic analysis of the time discretization takes place in a Banach space setting and uses a resolvent estimate for the associated elliptic operator. The methods are combined with finite element discretization in the spatial variables to yield fully discrete methods.

1. Introduction. For $-1 < \alpha < 1$, we shall consider numerical, particularly time discretization, methods for an initial-value problem of the form

$$\partial_t u + \partial_t^{-\alpha} A u = f(t), \quad \text{for} \quad t > 0, \quad \text{with} \quad u(0) = u_0, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, and where A is a sectorial linear operator in a complex Banach space $\mathcal{B}$.

In the applications we have in mind, A is a linear, second-order elliptic partial differential operator in some spatial variables (whose coefficients

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