JOURNAL OF COMMUTATIVE ALGEBRA
Volume 5, Number 1, Spring 2013

REGULARITY BOUNDS FOR BINOMIAL EDGE IDEALS

KAZUNORI MATSUDA AND SATOSHI MURAI

Dedicated to Professor Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. We show that the Castelnuovo-Mumford regularity of the binomial edge ideal of a graph is bounded below by the length of its longest induced path and bounded above by the number of its vertices.

1. Introduction. Let G be a simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$. The *binomial edge ideal* J_G of G, introduced by Herzog et al. [**4**] and Ohtani [**8**], is the ideal in the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field K, defined by

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \text{ is an edge of } G).$$

From an algebraic viewpoint, it is of interest to study relations between algebraic properties of J_G and combinatorial properties of G. In this note, we prove the following simple combinatorial bounds for the regularity of binomial edge ideals.

Theorem 1.1. *Let G be a simple graph on $[n]$, and let ℓ be the length of the longest induced path of G. Then*

$$\ell + 1 \leq \operatorname{reg}(J_G) \leq n.$$

2. A lower bound. In this section, we prove the lower bound in Theorem 1.1. Throughout the paper, we will use the standard terminologies of graph theory in [**2**].

We consider the $\mathbf{N}^n$-grading of S defined by $\deg x_i = \deg y_i = \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector of $\mathbf{N}^n$. Binomial edge ideals are $\mathbf{N}^n$-graded by definition. For an $\mathbf{N}^n$-graded S-module M and $\mathbf{a} \in \mathbf{N}^n$,

Received by the editors on September 16, 2012, and in revised form on December 6, 2012.
DOI:10.1216/JCA-2013-5-1-141