JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 2, Summer 2012

STABILITY OF QUASI-SOCLE IDEALS

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ABSTRACT. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\dim A > 0$. Let $\mathrm{G}(\mathfrak{m}) = \oplus_{n\geq 0}\mathfrak{m}^n/\mathfrak{m}^{n+1}$ be the associated graded ring of $\mathfrak{m}$. This paper explores quasi-socle ideals in A, i.e., ideals of the form $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ where Q is a parameter ideal. Goto, Sakurai, and the author have shown that the methods developed by Wang also work in the non Cohen-Macaulay case with some modification. The purpose of this paper is to solve a problem that has remained open. We will show that, if A is a generalized Cohen-Macaulay ring with $\operatorname{depth}\mathrm{G}(\mathfrak{m}) \geq 2$, then for each integer $q \geq 1$ one can find an integer $t = t(q) \gg 0$, depending upon q, such that $I^2 = QI$ for every parameter ideal Q contained in $\mathfrak{m}^t$, where $I = Q : \mathfrak{m}^q$. Therefore, the associated graded ring $\mathrm{G}(I) = \oplus_{n\geq 0}I^n/I^{n+1}$ of I is a Buchsbaum ring whenever A is Buchsbaum.

1. Introduction. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. This paper studies quasi-socle ideals, i.e., ideals of the form $I = Q : \mathfrak{m}^q$ $(q \geq 1)$ where Q is a parameter ideal in A. We are interested in determining when $I^2 = QI$, in which case we call I stable. To state the results, we need to first fix some notation and terminology.

For each $\mathfrak{m}$-primary ideal I in A, we denote by $\{\mathrm{e}_I^i(A)\}_{0\leq i\leq d}$ the Hilbert coefficients of A with respect to I. The Hilbert function of I is then given by the formula

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_I^d(A)$$

for all $n \gg 0$, where $\ell_A(M)$ denotes the length of the A-module M.

Let Q be a parameter ideal in A. We set $\mathbf{I}(Q) = \ell_A(A/Q) - e_Q^0(A)$. Then A is a Cohen-Macaulay ring if and only if $\mathbf{I}(Q) = 0$ for some (and

2010 AMS *Mathematics subject classification.* Primary 13H10, Secondary 13H15.
Keywords and phrases. Quasi-socle ideal, Cohen-Macaulay ring, Buchsbaum ring, associated graded ring, local cohomology, multiplicity.
Received by the editors on October 16, 2010, and in revised form on June 28, 2011.

DOI:10.1216/JCA-2012-4-2-269