JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 4, Winter 2011

ON POSITIVE AFFINE MONOIDS

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ABSTRACT. For numerical monoids S the length of the $k[S]$-module $k[\overline{S}]/k[S]$ is always finite. This is of course because the set of holes $H(S)$ is finite, a property that does not hold in general for positive affine monoids of higher rank. We examine here in a combinatorial fashion positive affine monoids S with $H(S)$ finite, or equivalently, positive affine monoids for which the length of $k[\overline{S}]/k[S]$ is finite. This class of monoids turns out to behave in some respects like numerical monoids. In particular we describe the maximal elements in certain posets whose elements are positive affine monoids. This description provides natural "higher dimensional" versions of familiar classes of numerical monoids such as the class of symmetric numerical monoids.

1. Preliminaries and notations.

An affine monoid $S = \langle s_1, \dots, s_n \rangle$ is a finitely generated sub-monoid of $\mathbf{Z}^r$ for some $r \in \mathbf{N}$, $r \geq 1$. We denote by $\operatorname{gp}(S)$ the group inside $\mathbf{Z}^r$ generated by S. Observe that every element $x \in \operatorname{gp}(S)$ can be written as $x = s - s'$ for some elements s and s' in S and that $\operatorname{gp}(S)$ is free of rank at most r. The rank of S, $\operatorname{rank}(S)$, is by definition the rank of $\operatorname{gp}(S)$. We assume all affine monoids S are embedded in $\mathbf{Z}^d$ where $d = \operatorname{rank}(S)$.

Our main concern will be positive affine monoids: an affine monoid is called positive if zero is the only element whose inverse in $\operatorname{gp}(S)$ also lies in S. A positive affine monoid $S = \langle s_1, \dots, s_n \rangle$ of rank d is isomorphic to an affine monoid T inside $\mathbf{N}^d$. Thus in the sequel all positive affine monoids S will be considered to be inside $\mathbf{N}^d$ where $d = \operatorname{rank}(S)$.

Assume $S = \langle s_1, \dots, s_n \rangle$ is a positive affine monoid of rank one such that $\gcd(s_1, \dots, s_n) = 1$. Then S is called a numerical monoid.

Any affine (respectively positive affine) monoid $S = \langle s_1, \dots, s_n \rangle$ gives rise to a cone (respectively pointed cone)

$$\mathbf{R}_{\geq 0} S = \mathbf{R}_{\geq 0}\{s_1, \dots, s_n\} = \{\lambda_1 s_1 + \cdots + \lambda_n s_n \ \lambda_i \in \mathbf{R}_{\geq 0}\}.$$

2010 AMS *Mathematics subject classification.* Primary 13A02, 20M14, 20M25, 20M50.
Received by the editors on October 4, 2010, and in revised form on August 1, 2011.
DOI:10.1216/JCA-2011-3-4-511