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TRIVARIATE MONOMIAL COMPLETE INTERSECTIONS AND PLANE PARTITIONS

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ABSTRACT. We consider the homogeneous components U_r of the map on $R = \mathbf{k}[x,y,z]/(x^A, y^B, z^C)$ that multiplies by $x+y+z$. We prove a relationship between the Smith normal forms of submatrices of an arbitrary Toeplitz matrix using Schur polynomials and use this to give a relationship between Smith normal form entries of U_r. We also give a bijective proof of an identity proven by Li and Zanello equating the determinant of the middle homogeneous component U_r when $(A,B,C) = (a+b, a+c, b+c)$ to the number of plane partitions in an $a \times b \times c$ box. Finally, we prove that, for certain vector subspaces of R, similar identities hold relating determinants to symmetry classes of plane partitions, in particular classes 3, 6 and 8.

1. Introduction. For a commutative ring $\mathbf{k}$ and positive integers A, B, C, consider the trivariate monomial complete intersection $R = \mathbf{k}[x,y,z]/(x^A, y^B, z^C)$. This carries a standard grading in which x, y, z each have degree one and decomposes as a direct sum $R = \bigoplus_{r=0}^{e} R_r$ where $e := A+B+C-3$, and each homogeneous component $R_r \cong \mathbf{k}^{h(r)}$, where $h(r)$ denotes the size of the set B_r consisting of all monomials of total degree r in x, y, z which are nonzero in R. It is easily seen that $(h(0), h(1), \ldots, h(e))$ is a symmetric unimodal sequence. Furthermore, it is known that the maps

$$U_r : R_r \stackrel{\cdot(x+y+z)}{\longrightarrow} R_{r+1}$$

have $U_{e-r}^t = U_r$, and that U_r is injective for $0 \le r \le \lfloor (e-1)/2 \rfloor$ when working with $\mathbf{k} = \mathbf{Z}$ or $\mathbf{Q}$ (or, in fact, with any field of characteristic zero).

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