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ON IDEAL EXTENSIONS OF IDEAL COMPLEMENTS

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ABSTRACT. In this note we give negative answers to a conjecture of Tomas Sauer. Specifically we prove that there exists an ideal $K \subset \mathbf{C}[x, y]$ that complements the space of polynomials of degree 3 such that no ideal containing K complements the space of polynomials of degree 2. We also give a characterization of zero-dimensional radical ideals in terms of extensions of ideal complements.

1. Introduction. Let $\mathbf{C}[\mathbf{x}] := \mathbf{C}[x_1, \dots , x_d]$ stand for the algebra of polynomials in d variables with complex coefficients and $\mathbf{C}_{\le N}[\mathbf{x}]$ denote the linear subspace of $\mathbf{C}[\mathbf{x}]$ of polynomials of degree at most N. For an ideal $J \subset \mathbf{C}[\mathbf{x}]$ we use $\mathcal{V}(J)$ to denote the affine variety associated with this ideal.

The extensions of ideals complements is the object of investigations related to the multivariate Lagrange and Hermite interpolation (cf. [**3**–**5**]). Paraphrased, a result of Sauer and Xu [**5**] shows that every radical ideal that complements $\mathbf{C}_{\le N}[\mathbf{x}]$ in $\mathbf{C}[\mathbf{x}]$ can be extended to a (zero-dimensional, radical) ideal that complements $\mathbf{C}_{\le N-1}[\mathbf{x}]$. In other words, if $K \subset \mathbf{C}[\mathbf{x}]$ is a radical ideal such that

$$\mathbf{C}[\mathbf{x}] = \mathbf{C}_{\le N}[\mathbf{x}] \oplus K, \tag{1.1}$$

then there exists a radical ideal $J \supset K$ such that

$$\mathbf{C}[\mathbf{x}] = \mathbf{C}_{\le N-1}[\mathbf{x}] \oplus J. \tag{1.2}$$

Based on this result as well as some further evidence (cf. [**4**]), Sauer [**3**] made the following conjecture:

Conjecture 1.1. *If K is an arbitrary ideal in $\mathbf{C}[\mathbf{x}]$ satisfying* (1.1), *then there exists an ideal $J \supset K$ satisfying* (1.2).

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