ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 6, 2007

TORSION THEORIES FOR ALGEBRAS OF AFFILIATED OPERATORS OF FINITE VON NEUMANN ALGEBRAS

LIA VAŠ

ABSTRACT. The dimension of any module over an algebra of affiliated operators $\mathcal{U}$ of a finite von Neumann algebra $\mathcal{A}$ is defined using a trace on $\mathcal{A}$. All zero-dimensional $\mathcal{U}$-modules constitute the torsion class of torsion theory $(\mathbf{T}, \mathbf{P})$. We show that every finitely generated $\mathcal{U}$-module splits as the direct sum of torsion and torsion-free part. Moreover, we prove that the theory $(\mathbf{T}, \mathbf{P})$ coincides with the theory of bounded and unbounded modules and also with the Lambek and Goldie torsion theories. Lastly, we use the introduced torsion theories to give the necessary and sufficient conditions for $\mathcal{U}$ to be semi-simple.

1. Introduction. A finite von Neumann algebra proves to be an interesting structure both for operator theorists and for those working in geometry or algebraic K-theory. One of the reasons is that a finite von Neumann algebra $\mathcal{A}$ comes equipped with a normal and faithful trace that enables us to define the dimension not just of a finitely generated projective module over $\mathcal{A}$ but also of arbitrary $\mathcal{A}$-module.

Moreover, $\mathcal{A}$ mimics the ring $\mathbf{Z}$ in such a way that every finitely generated module is a direct sum of a torsion and torsion-free part. The dimension faithfully measures the torsion-free part and vanishes on the torsion part. $\mathcal{A}$ has nice ring-theoretic properties: it is semi-hereditary, i.e., every finitely generated submodule of a projective module is projective, and an Ore ring. The fact that $\mathcal{A}$ is an Ore ring allows us to define the classical ring of quotients denoted $\mathcal{U}$. Besides this algebraic definition, it turns out that, within the operator theory, $\mathcal{U}$ can be defined as the algebra of affiliated operators.

2000 AMS *Mathematics subject classification.* Primary 16W99, 46L99, 16S90.
Keywords and phrases. Finite von Neumann algebra, algebra of affiliated operators, torsion theories.
Part of the results were obtained during the time the author was at the University of Maryland, College Park. The author was supported by NSF grant DMS9971648 at that time.
Received by the editors on March 3, 2004, and in revised form on July 24, 2005.