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FLAT FACES IN PUNCTURED TORUS GROUPS

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1. Introduction. One of the intriguing constructions which arises from the use of the Minkowski space model for hyperbolic space is that of the *canonical triangulation* of a one cusped manifold. We briefly recall its definition in the dimensions relevant to this paper, namely, $n = 2, 3$ (a somewhat more complete description is included below). It follows from results of Epstein and Penner [**1**] that Jorgensen's lemma implies that the action of a finite covolume group on a lightlike vector corresponding to the cusp forms a discrete subset of the light cone. One can take the convex hull in Minkowski space of this set and project into hyperbolic space to obtain an equivariant cellulation (generically a triangulation) which descends to the manifold. In the case that the hyperbolic manifold in question only has one cusp, the only ambiguity is a possibly scaling in the initial choice of lightlike vector representing this cusp. Two scaled convex hulls project to the same triangulation.

Despite the elegant nature of this construction, it is not very well understood, and several authors have attempted to identify this triangulation in special cases, see Weeks [**11**], Sakuma [**7**], Lackenby [**2**]. This paper attempts to understand the convex hull in a somewhat simpler setting, namely in the case of punctured torus groups acting on $\mathbf{H}^2$. Even here the answers turn out to be rather subtle. We begin by proving a condition (originally sketched by Thurston) which guarantees that once we have taken the hull of enough points, this will indeed form part of the canonical convex hull. As an initial step, we show:

Theorem 3.1.2. *Local boundary convexity of a polyhedron implies boundary convexity.*

With this theorem in hand, we can begin to analyze the convex hulls of punctured torus groups. Here we use a description first introduced in [**3**] of punctured torus groups as $\Delta(u^2, 2\tau)$; this is explained in subsection 3.1. While not a parametrization (different values may give

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