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UPPER BOUNDS FOR UNITARY PERFECT NUMBERS AND UNITARY HARMONIC NUMBERS

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ABSTRACT. We prove the following two theorems: (1) If N is a unitary perfect number with k distinct prime factors, then $N < 2^{2^k}$. (2) If N is a unitary harmonic number with k distinct prime factors, then $N < (2^{2^k})^k$.

1. Introduction. Let σ_j be the divisor function defined by

$$\sigma_j(N) = \sum_{d \mid N} d^j.$$

This function is multiplicative, that is, $\sigma_j(ab) = \sigma_j(a)\sigma_j(b)$ if $(a, b) = 1$. A positive integer N is said to be a *perfect number* if $\sigma_1(N) = 2N$. It is well known that an even perfect number has a form $2^{p-1}(2^p - 1)$ with $2^p - 1$ prime. As of October, 2006, 44 even perfect numbers are known (for the newest information, see the web site of GIMPS: `http://www.mersenne.org/prime.htm`). It is still open whether or not odd perfect numbers (OPNs) exist; however, many conditions for their existence are known. For example, Brent, Cohen and te Riele [**1**] showed that OPNs must be greater than 10^{300}. Suppose that N is an OPN with k distinct prime factors. Dickson [**5**] showed that, for a fixed positive integer k, there exist only finitely many such N. Moreover, it was shown by Hagis [**7**] and Chein [**2**] independently that k must be greater than 7. Pomerance [**13**] showed that $N < (4k)^{(4k)2^{k^2}}$, and this bound was improved by Heath-Brown [**9**] to 4^{4^k}, by Cook [**4**] to D^{4^k} with $D = (195)^{1/7} \approx 2.12$, by Nielsen [**10**] to 2^{4^k}.

Subbarao and Warren [**15**] introduced the concept of *unitary perfect numbers* (UPNs). A positive integer d is said to be a unitary divisor of N if $d \mid N$ and $(d, N/d) = 1$. So we define the unitary divisor function

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