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ON THE SUM FORMULA FOR MULTIPLE q-ZETA VALUES

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ABSTRACT. Multiple q-zeta values are a one-parameter generalization (in fact, a q-analog) of the multiple harmonic sums commonly referred to as multiple zeta values. These latter are obtained from the multiple q-zeta values in the limit as $q \to 1$. Here, we discuss the sum formula for multiple q-zeta values, and provide a direct, self-contained proof. As a consequence, we also derive a q-analog of Euler's evaluation of the double zeta function $\zeta(m, 1)$.

1. Introduction. Sums of the form

$$\zeta(n_1, n_2, \dots, n_r) := \sum_{k_1 > k_2 > \cdots > k_r > 0} \prod_{j=1}^{r} \frac{1}{k_j^{n_j}} \tag{1}$$

have attracted increasing attention in recent years, see e.g., [**1**–**4, 6**–**8, 10, 11**]. The survey articles [**5, 19, 20**] provide an extensive list of references. Here and throughout, $n_1, \dots, n_r$ are positive integers with $n_1 > 1$, and we sum over all positive integers $k_1, \dots, k_r$ satisfying the indicated inequalities. Note that, with positive integer arguments, $n_1 > 1$ is necessary and sufficient for convergence. The sums (1) are sometimes referred to as Euler sums, because they were first studied by Euler [**12**] in the case $r = 2$. In general, they may be profitably viewed as instances of the multiple polylogarithm [**2, 5, 18**], and are now more commonly referred to as multiple zeta values, reducing to the Riemann zeta function in the case $r = 1$. The present author introduced a q-analog of (1) in [**9**] as

$$\zeta[n_1, n_2, \dots, n_r] := \sum_{k_1 > k_2 > \cdots > k_r > 0} \prod_{j=1}^{r} \frac{q^{(n_j - 1)k_j}}{[k_j]_q^{n_j}}, \tag{2}$$

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