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THE STEREOGRAPHIC PROJECTION IN BANACH SPACES

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ABSTRACT. We give a new and direct proof of the fact that, in any infinite dimensional Banach space, the unit sphere minus any one point is homeomorphic to a closed hyperplane. The proof involves L-structures and geometric concepts as, for instance, rotund, smooth and exposed points.

1. Preliminaries and background. It is well known [**1**] that the Euclidean unit sphere $\mathcal{S}^n$ minus any one point is homeomorphic to $\mathbf{R}^n$; this homeomorphism is known as the *stereographic projection*. This stereographic projection can be generalized to infinite dimensional spaces or, more particularly, to infinite dimensional real Banach spaces. This is the aim of this paper, to give a new and direct proof of this result, i.e., that the unit sphere minus any one point is homeomorphic to a closed hyperplane in any real Banach space.

On the other hand, to establish homeomorphisms between unit balls and/or unit spheres in a Banach space, it suffices to consider isomorphisms of Banach spaces. In other words, if X and Y are isomorphic Banach spaces, and $T : X \to Y$ is an isomorphism, then the mapping $T_B : \mathcal{B}_X \to \mathcal{B}_Y$, given by

$$\begin{cases} T_B : \mathcal{B}_X \longrightarrow \mathcal{B}_Y \\ x \longmapsto T_B x, \end{cases}$$

where

$$T_B x = \begin{cases} Tx/\|Tx\| \cdot \|x\| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

and where $\mathcal{B}_X$ is the unit ball of X, is an homeomorphism whose restriction to $\mathcal{S}_X$ (the unit sphere of X) induces an homeomorphism between $\mathcal{S}_X$ and $\mathcal{S}_Y$. This fact will be used later on to establish the main result. Next, let us recall the definition of the L^2*-summand vector*, see [**3**].

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