ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 2003

STEEPEST DESCENT ON A UNIFORMLY CONVEX SPACE

MOHAMAD M. ZAHRAN

1. Introduction. The idea of steepest descent and how it is of use to find zeros or critical points of nonnegative C^2 functions defined on Hilbert spaces is extensively presented in [**7, 8, 9**] and [**14**]. The main objective of this paper is to generalize parts of these references to many problems of interest or set naturally in the uniformly convex space setting. We are concerned here with numerical solutions of differential equations that fit into the uniformly convex Sobolev spaces $H^{1,p}(\Omega)$ for $p > 2$ and $\Omega \subset R^n$ and do not fit conveniently into the Hilbert space $H^{1,2}(\Omega)$. A good example of this situation is the diffusion problem of the form

$$-\Delta y + F'(y) = 0, \quad y \in H^{1,p}(\Omega)$$

where F is a polynomial function. Let

$$\varphi(y) = \frac{1}{2}\int_\Omega y_1^2 + y_2^2 + y_3^2 + F(y), \tag{1}$$

where y_i is the partial derivative with respect to the ith variable.

We seek $y \in H^{1,p}(\Omega)$ so that $\varphi'(y)h = 0$ for all $h \in H^{1,p}(\Omega)$. To do this, note that if $y \in H^{2,p}(\Omega)$,

$$\begin{aligned}\varphi'(y)h &= \int_\Omega h_1y_1 + h_2y_2 + h_3y_3 + F'(y)h \\ &= \int_{\partial\Omega} h\frac{\partial y}{\partial n} + \int_\Omega (-(y_{11} + y_{22} + y_{33}) + F'(y))h = 0.\end{aligned}$$

This implies that $-\Delta y + F'(y) = 0$ with $\partial y/\partial n = 0$ on the boundary $\partial\Omega$, where n is the outward normal of Ω.

φ in (1) will be well defined if p is chosen so that $F(y) = y^8 \in L_1(\Omega)$ for $y \in H^{1,p}(\Omega)$ and $\Omega \subset R^3$. By the Sobolev embedding theorem in [**1**], it is sufficient to choose p so that $8 \leq 3p/3 - p$. The best choice

Received by the editors on May 21, 2001.