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ON SUMS OF TWO SQUARES AND SUMS OF TWO TRIANGULAR NUMBERS

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ABSTRACT. For each integer $n \geq 0$, $r_2(n)[t_2(n)]$ denotes the number of representations of n by sums of two squares (two triangular numbers). Similarities and differences of the two functions r_2 and t_2 are described, with the major contribution being an apparently new recursive determination of t_2.

1. Introduction. We begin with a definition.

Definition 1.1. As usual, $\mathbf{P} := \{1, 2, 3, \dots\}$, $\mathbf{N} := \mathbf{P} \cup \{0\}$ and $\mathbf{Z} := \{0, \pm 1, \pm 2, \dots\}$. Then for each $n \in \mathbf{N}$,

$$r_2(n) := |\{(x, y) \in \mathbf{Z}^2 \mid n = x^2 + y^2\}|,$$
$$t_2(n) := |\{(x, y) \in \mathbf{N}^2 \mid n = x(x+1)/2 + y(y+1)/2\}|.$$

Also for each $n \in \mathbf{P}$ and each $i \in \{1, 3\}$,

$$d_i(n) := \sum_{\substack{d \mid n \\ d \equiv i \pmod 4}} 1.$$

That the functions r_2 and t_2 are closely related is revealed by the next two theorems and their obvious corollary.

Theorem 1.2 (Jacobi). *For each $n \in \mathbf{P}$,*

$$r_2(n) = 4\{d_1(n) - d_3(n)\}.$$

(*Of course,* $r_2(0) = 1$.)

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