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SOLUTION OF A PROBLEM ABOUT SYMMETRIC FUNCTIONS

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ABSTRACT. Let $a > b > c$ be positive integers with $(a, b, c) = 1$. Then the field $\mathbf{Q}(X^a + Y^a, X^b + Y^b, X^c + Y^c)$ is the field of all symmetric rational functions in X, Y over $\mathbf{Q}$. This solves a conjecture made by Mead and Stein.

Let X, Y be independent indeterminates and, for a positive integer m, let

$$N_m = N_m(X, Y) = X^m + Y^m$$

be the Newton symmetric power of order m. In the recent paper [**2**], the authors calculate the degree $[S : \mathbf{Q}(N_a, N_b)]$, where S is the field of all symmetric rational functions in X, Y with rational coefficients. They also raise a few conjectures on the fields $\mathbf{Q}(N_a, N_b, N_c)$. The purpose of the present paper is to prove their main Conjecture 1, which we state as the following.

Theorem 1. *If $a > b > c$ are distinct positive integers with $(a, b, c) = 1$, then the functions N_a, N_b, N_c generate S over $\mathbf{Q}$.*

In [**2**] the authors also state a conjecture (see Conjecture 4 of Section 3) about the minimal degree d of a polynomial relation satisfied by N_a, N_b, N_c where, by degree of a monomial $N_a^i N_b^j N_c^k$, they mean $ai + bj + ck$. At the end of the paper we shall show how our Theorem 1 implies a strong form of their conjecture, namely,

Theorem 2. *Assumptions being as in Theorem 1, we have $d = abc/2$ if abc is even and $d = (a - 1)bc/2$ otherwise.*

Proof of Theorem 1. To start with, we show that it is sufficient to prove the analogous statement with $\mathbf{Q}$ replaced by its algebraic closure

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