ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 2003

ON A THEOREM OF BANACH AND KURATOWSKI AND K-LUSIN SETS

TOMEK BARTOSZYŃSKI AND LORENZ HALBEISEN

ABSTRACT. In a paper of 1929, Banach and Kuratowski proved—assuming the continuum hypothesis—a combinatorial theorem which implies that there is no non-vanishing σ-additive finite measure μ on $\mathbf{R}$ which is defined for every set of reals. It will be shown that the combinatorial theorem is equivalent to the existence of a K-Lusin set of size $2^{\aleph_0}$ and that the existence of such sets is independent of ZFC $+ \neg$CH.

0. Introduction. In [**1**], Stefan Banach and Kazimierz Kuratowski investigated the following problem in measure theory:

Problem. *Does there exist a non-vanishing finite measure* μ *on* $[0,1]$ *defined for every* $X \subseteq [0,1]$*, which is* σ*-additive and such that for each* $x \in [0,1]$*,* $\mu(\{x\}) = 0$*?*

They showed that such a measure does not exist if one assumes the continuum hypothesis, denoted by CH. More precisely, assuming CH, they proved a combinatorial theorem [**1**, Théorème II] and showed that this theorem implies the nonexistence of such a measure. The combinatorial result is as follows:

Banach-Kuratowski theorem. *Under the assumption of* CH*, there is an infinite matrix* $A^i_k \subseteq [0,1]$ *(where* $i, k \in \omega$*) such that*:

(i) *For each* $i \in \omega$*,* $[0,1] = \cup_{k\in\omega} A^i_k$*.*

(ii) *For each* $i \in \omega$*, if* $k \neq k'$*, then* $A^i_k \cap A^i_{k'} = \varnothing$*.*

2000 AMS *Mathematics Subject Classification.* 03E35, 03E17, 03E05.
Key words and phrases. Combinatorial set theory, continuum hypothesis, Lusin sets, consistency results, cardinal characteristics.
First author partially supported by NSF grant DMS 9971282 and Alexander von Humboldt Foundation.
Received by the editors on February 16, 2001, and in revised form on August 20, 2001.