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THE RELATIVE CONSISTENCY OF A "LARGE CARDINAL" PROPERTY FOR ω_1

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ABSTRACT. There are two current methods for obtaining "Large Cardinal" properties for ω_1 and other small cardinals in the absence of AC (the Axiom of Choice): one is to work within ZF + AD (the Axiom of Determinateness); the other method is to prove the consistency of the desired property for ω_1 in ZF, assuming the consistency of ZF + AC + whatever other axiom is required. Both approaches yield ω_1 measurable. Using a straightforward modification of Jech's result, which showed ω_1 can be measurable, and assuming ZF + AC + there is a measurable cardinal, we prove ω_1 can be huge and, assuming ZF + AC +, there is a huge cardinal. In view of the well-known result that ω_1 is measurable in ZF + AD, our result should heighten interest in the problem of showing that ω_1 is huge in ZF + AD + DC, which is yet unresolved.

Introduction. A cardinal κ is called huge if $\kappa > \omega$ and, for some $\lambda > \kappa$, there is a κ-complete, fine, normal ultrafilter U on the field of all subsets of precisely the subsets of λ of order type κ. In ZF a huge cardinal is measurable. In fact, if κ is huge and $\lambda > \kappa$ is the necessary cardinal from the definition above, then κ is λ-supercompact (see [**5**]). With regard to the strength of huge cardinals for providing consistency results, it is known that, from the assumption of a huge cardinal in ZFC, the consistency of Vopěnka's principle follows, from which the existence of extendible cardinals and hence supercompact cardinals follows (see [**3**]and [**4**]). Indeed, in ZFC a huge cardinal is a very large cardinal.

It is interesting to investigate what the possibilities are in the absence of choice for small cardinals, in the partial ordering of cardinals in ZF, to possess "Large Cardinal" properties.

Currently, there are two major techniques for obtaining "Large Cardinal" properties in the absence of choice: one is to work within ZF + AD (the Axiom of Determinateness); the other is to prove the consistency of the desired property in ZF, assuming the consistency of ZF + AC +

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