ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

THE METRIZATION OF UPPER LIMIT TOPOLOGIES

WARREN M. KRUEGER

1. Introduction. A fruitful source of examples in general topology arises from certain linearly ordered sets endowed with the upper limit topology (e.g., the reals and various spaces derived from the ordinal space $[0, \Omega]$). Their usefulness stems in part from the fact that, although these particular spaces are not metrizable, each possesses many properties necessary for metrization.

In this note, we deduce the non-metrizability of these spaces from a result which characterizes metrizability of the upper limit topology on an arbitrary totally ordered set. The method is both elementary and direct—elementary in that it uses only the definition of the order topology and simple topological properties and direct in that it does not require the use of auxiliary spaces such as the Cartesian square.

The author wishes to thank Art Kruse for his perspicacity in general and for his suggestions regarding Theorem 4 and Corollary 5 in particular.

2. Preliminaries. Let $(X, <)$ be a totally ordered set. The upper limit topology induced by $<$ is the one, a basis for which is

$$\{(a, b] \mid a, b \in X, a < b\}.$$

Following Dugundji [**1**, p. 66], we call the resulting space X_u. Suppose d is a metric on the ordered set X; its topology need not be that of X_u.

For $\epsilon > 0$ and $y \in X$, we adopt the notation $N_d(y, \epsilon)$ for

$$\{x \in X \mid d(y, x) < \epsilon\}.$$

Now define $f : X \to \mathbf{R} \cup \{\infty\}$ by the rule

$$f(y) = \begin{cases} \text{glb}\{d(x, y) \mid y < x\}, & \text{if } (y, \infty) \neq \varnothing, \\ \infty, & \text{otherwise.} \end{cases}$$

Received by the editors on April 1, 1985 and in revised form on March 3, 1986.