ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 1, Winter 1990

AZUMAYA ALGEBRAS WHICH ARE NOT SMASH PRODUCTS

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Let R be a commutative ring, let H be an R-Hopf algebra (always with antipode), finitely generated and projective as an R-module, and let $H^* = \operatorname{Hom}_R(H, R)$ be the dual Hopf algebra. Let $\operatorname{Gal}(H)$ denote the set of isomorphism classes (as R-algebras and H-modules) of Galois H-extensions (that is, Galois H^*-objects, in the sense of [**3**, §7]). Let $\operatorname{Az}(R)$ denote the set of isomorphism classes (as R-algebras) of Azumaya R-algebras. Gamst and Hoechsmann [**15**] showed that if S is a Galois H-extension and T a Galois H^*-extension, then the smash product $S \# T$ is an Azumaya R-algebra, so yields a map

$$\# \ : \operatorname{Gal}(H) \times \operatorname{Gal}(H^*) \to \operatorname{Az}(R)$$

given by $[S] \times [T] \mapsto [S \# T]$.

The smash product generalizes the cyclic crossed product. As we will show in §2 below, for rank 2 Hopf algebras, Sweedler's crossed product based on Hopf algebra cohomology also is a special case of the smash product.

Let $\operatorname{Br}(R)$ be the Brauer group of R, and $\{ \ \}$ denote the class map, $\operatorname{Az}(R) \to \operatorname{Br}(R)$. If H is commutative and cocommutative, then $\operatorname{Gal}(H)$ and $\operatorname{Gal}(H^*)$ are abelian groups, and $\{ \# \}$ is bilinear. (See [**15**] for an interpretation of $\{ \# \}$ as a cup product map.) In the special case where R is a field containing $1/n$ and a primitive n^{th} root of unity and $H = RG$, G cyclic of order n, then $H \cong H^*$, $\operatorname{Gal}(H) \cong U(R)/U(R)^n$ and the smash product map $\{ \# \}$ specializes to the norm residue map which Merkurjev and Suslin showed maps onto the n-torsion part of the Brauer group.

Thus, over number fields, every Azumaya algebra is isomorphic to a smash product, and over many fields every Azumaya algebra is at least similar to a product of smash products.

Received by the editors on February 25, 1987 and in revised form on April 30, 1987.