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ENTROPY OF CERTAIN NONCOMMUTATIVE SHIFTS

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ABSTRACT. The entropy of the noncommutative shift of the hyperfinite II_1-factor associated with a sequence of Jones' projections is computed.

0. Introduction. By the work of V.F.R. Jones [**3**] and M.Pimsner-S. Popa [**4**], for any real number λ in the set $(0, 1/4] \cup \{(\sec^2 \pi/n)/4 : n \geq 3\}$, we can find a doubly infinite sequence of projections $\{e_i : i \in \mathbf{Z}\}$ in the hyperfinite II_1-factor R such that the e_i's generate R as a von Neumann algebra and satisfy the conditions: (a) $e_i e_j e_i = \lambda e_i$ if $|i - j| = 1$; (b) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$; and (c) $\mathrm{tr}(w e_n) = \lambda \mathrm{tr}(w)$, if w is a word on 1 and $\{e_i : i < n\}$. Here tr is the unique normal normalized trace on R. Then $e_i \to e_{i+1}$ defines an ergodic automorphism Θ_λ of R [**4**, §5]. The Connes-Stormer entropy [**2**] of $\Theta_\lambda, H(\Theta_\lambda)$, was computed by Pimsner-Popa in [**4**, §5], among other things. The results are (i) $H(\Theta_\lambda) = (-\ln \lambda)/2$ if $\lambda = (\sec^2 \pi/n)/4, n \geq 3$, and (ii) $H(\Theta_\lambda) = \eta(t) + \eta(1-t)$ if $\lambda < 1/4$, where η is the function $\eta(x) = -x \ln x, x \in \mathbf{R}$, and where $t(1-t) = \lambda$. The case $\lambda = 1/4$ is left open [**4**, p. 92]. In this note we prove $H(\Theta_{1/4}) = \ln 2$, thus completing the circle. Pimsner-Popa showed that, when $\lambda < 1/4$, the Θ_λ are just the noncommutative Bernoulli shifts of Connes-Størmer and Krieger [**2**] with weights $\{t, 1-t\}$, and thus obtained the entropy by the computation in [**2**]. Of course, their approach provides more results than merely the entropy. However, it seems worthwhile to give a direct computation of the entropy. We include such a computation hereon.

Our computation is based on the explicit knowledge of the structure of the finite dimensional algebras $A_n = \{e_1, e_2, \dots, e_n\}''$ provided by Jones [**3**].

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1. Preliminaries. Let $\{e_i : i \in \mathbf{Z}\}$ be a sequence of projections satisfying the conditions (a), (b), (c) of §0 for some λ in the Jones

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