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RESTRICTIONS OF ESSENTIALLY NORMAL OPERATORS

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Let $\mathcal{H}$ be a complex, infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{C}$ denote the ideal of all compact operators in $\mathcal{L}(\mathcal{H})$, and let π denote the natural quotient map of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{C}$. For T in $\mathcal{L}(\mathcal{H})$, let $\tilde{T} = \pi(T)$. Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is called *essentially normal* if $\tilde{T}$ is normal, or, equivalently, if the self-commutator $T^*T - TT^*$ is compact. Let T be an operator in $\mathcal{L}(\mathcal{H})$ that is unitarily equivalent to the bilateral shift of infinite multiplicity. There exists an invariant subspace $\mathcal{M}$ for T such that $T|\mathcal{M}$ is unitarily equivalent to the unilateral shift of infinite multiplicity. Note that T is essentially normal (it's normal), but $T|\mathcal{M}$ is not essentially normal. Thus the restriction of an essentially normal operator to an invariant subspace is not necessarily essentially normal.

Recall that an operator S in $\mathcal{L}(\mathcal{H})$ is said to be subnormal if it has a normal extension. Bunce and Deddens proved in [**2**] that an operator S in $\mathcal{L}(\mathcal{H})$ is *subnormal* if and only if, for each $B_0, B_1, \ldots, B_n$ in $C^*(S)$, the C^*-algebra generated by S and $1_{\mathcal{H}}$, (or equivalently in $\mathcal{L}(\mathcal{H})$),

$$\sum_{k=0}^{n}\sum_{j=0}^{n} B_j^* S^{*k} S^j B_k \geq 0_{\mathcal{H}}. \tag{1}$$

(See also [**4**]). This characterization of a subnormal operator is completely algebraic, and Bunce has used it to define a subnormal element of an abstract C^*-algebra [**1**]. Accordingly, we shall say that an element S of the Calkin algebra is *subnormal* if, for each $B_0, B_1, \ldots, B_n$ in $\mathcal{L}(\mathcal{H})/\mathcal{C}$, (1) holds. An operator S in $\mathcal{L}(\mathcal{H})$ is said to be *essentially subnormal* if $\tilde{S}$ is a subnormal element of the Calkin algebra. Observe that each essentially normal operator is essentially subnormal. We noticed above that the restriction of an essentially normal operator to an invariant subspace is not necessarily essentially normal. However, it follows readily that the restriction of an essentially normal operator to an

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