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ON MINIMAL UPPER SEMICONTINUOUS COMPACT-VALUED MAPS

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1. Introduction. In what follows, X and Y are Hausdorff topological spaces, and the term *map* is reserved for set-valued mappings. Also, for $x \in X$ and $y \in Y$, $\mathcal{U}(x)$ and $\mathcal{V}(y)$ are always used to denote a base of neighborhoods of x in X and y in Y, respectively. If $F : X \to Y$ is a (set-valued!) map, then

$$\mathrm{Gr}\,(F) = \{(x, y) \in X \times Y : y \in F(x)\}$$

is the *graph* of F.

Given two maps $F, G : X \to Y$, we write $G \subset F$ and say that G *is contained in* F if $G(x) \subset F(x)$ for every x in X; equivalently, if $\mathrm{Gr}\,(G) \subset \mathrm{Gr}\,(F)$. The relation of containment being a partial order in the family of all maps (with domain X and range Y), if a set $\mathcal{F}$ of maps is specified, we can look for maps which are minimal elements of $(\mathcal{F}, \subset)$.

A map $F : X \to Y$ is *upper semincontinuous at* a point $x \in X$ (*usc at* x) if, for every open set V containing $F(x)$, there exists $U \in \mathcal{U}(x)$ such that

$$F(U) = \cup\{F(u) : u \in U\} \subset V.$$

F is *upper semicontinuous* (*usc*) if it is usc at each point of X. We say, shortly, that a map F is *usco* if it is usc and takes nonempty compact values. Finally, a map F is said to be *minimal usco* if it is a minimal element in the family of all usco maps (with domain X and range Y); that is, if it is usco and does not contain properly any other usco map from X into Y. (See [**5**] for references.)

Historically, minimal usco maps seem to have appeared first in complex analysis (in the second half of the 19th century), in the form of a bounded holomorphic function and its "cluster sets," see, e.g., [**3**]. Starting with a 1982 paper of Christensen [**2**], a series of "multi-valued Namioka theorems" has been discovered (see [**9, 4**]). These theorems tell us that, under unexpectedly general assumptions about X and Y, a minimal usco map $F : X \to Y$ reduces to a (point-valued) function f on a

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