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EVOLUTIONARILY STABLE STRATEGIES DEPENDING ON POPULATION DENSITY

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ABSTRACT. The concept of evolutionarily stable strategies is extended to include density dependence. Dynamical stability is shown to follow for two-strategy games and for symmetric payoff matrices. It is conjectured that stability also results for general multi-strategy games.

1. The dynamical model. The theory of evolutionarily stable strategies (ESS) has been used primarily to predict long-term outcomes of selection models where an individual's fitness depends on the frequency of strategy-types in the population but not on the total population size (density). The purpose of this paper is to show that suitably-modified ESS concepts remain relevant for stability of more general dynamical models that include both frequency and density dependence.

Suppose each individual in the population uses one of the pure strategies $S_1, \ldots, S_n$. Let N_i be the number of S_i-users at time t and $N = \Sigma N_i$ be the density. Then

$$S = [s_1, \ldots, s_n],$$

where $s_i = N_i/N$ is a probability vector whose component s_i is the frequency of S_i-users in the population. We refer to S as the mean strategy of the population and to (S, N) as the state of the population at time t.

Assume that the fitness $F_i(S, N)$ of an S_i-individual depends only on the state (S, N), and, furthermore, that this fitness is linear in the components of S. (This linearity can also be obtained by linearizing the dynamics (1.2) about a point.) Then

$$F_i(S, N) = \sum_{k=1}^{n} A_{ik}(N) s_k, \tag{1.1}$$

where we call the $n \times n$ matrix $A(N)$ the density-dependent payoff matrix with entry $A_{ik}(N)$ thought of as the gain (or loss if A_{ik} is