ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 4, Fall 1990

COMPOUND MATRICES AND ORDINARY DIFFERENTIAL EQUATIONS

JAMES S. MULDOWNEY

This paper is dedicated to the memory of Geoffrey Butler

ABSTRACT. A survey is given of a connection between compound matrices and ordinary differential equations. Some typical linear results are presented. For nonlinear autonomous systems, a criterion for orbital asymptotic stability of a closed trajectory given by Poincaré in two dimensions is extended to systems of any finite dimension. A criterion of Bendixson for the nonexistence of periodic solutions of a two dimensional system is also extended to higher dimensional systems.

1. Introduction. Let X be any $n \times m$ matrix of real or complex numbers, and let $x^{j_1 \ldots j_k}_{i_1 \ldots i_k}$ denote the minor of X determined by the rows $(i_1, \ldots, i_k)$ and the columns $(j_1, \ldots, j_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, $1 \le j_1 < j_2 < \cdots < j_k \le m$. The k-th multiplicative compound $X^{(k)}$ of X is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries, written in lexicographic order, are $x^{j_1 \ldots j_k}_{i_1 \ldots i_k}$. In particular, when X is $n \times k$ with columns $x^1, \ldots, x^k$, then $X^{(k)}$ is the exterior product $x^1 \wedge \cdots \wedge x^k$ represented as a column vector. The term "multiplicative" is used since the Binet-Cauchy Theorem [**13**, p. 17] states that

$$(AB)^{(k)} = A^{(k)} B^{(k)} \tag{1.1}$$

for any matrices A and B of dimension consistent with the multiplication. An immediate consequence of (1.1) is that, for any nonsingular $n \times n$ matrix X, $(X^{(k)})^{-1} = (X^{-1})^{(k)}$, since $I^{(k)}$ is clearly the $\binom{n}{k} \times \binom{n}{k}$ identity, if I is the $n \times n$ identity.

When $m = n$, the k-th additive compound $X^{[k]}$ of X is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined by

$$X^{[k]} = D(I + hX)^{(k)}|_{h=0}, \tag{1.2}$$

Research supported by the Natural Sciences and Engineering Research Council of Canada under Grant NSERC A7197.