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TWO FAMILIES OF ORTHOGONAL POLYNOMIALS RELATED TO JACOBI POLYNOMIALS

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Dedicated to Wolfgang Thron on his 70th birthday

ABSTRACT. A family of orthogonal polynomials that generalize Jacobi polynomials is introduced. The exceptional case $\alpha+\beta=0$ of Jacobi polynomials is investigated.

1. Introduction. The Jacobi polynomials $\{P_n^{\alpha,\beta}(x)\}$ satisfy the three term recurrence relation

(1.1)
$$\begin{aligned}2(n+1)&(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{\alpha,\beta}(x)\\ &=(2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+(\alpha^2-\beta^2)]\\ &\quad\cdot P_n^{\alpha,\beta}(x)-2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{\alpha,\beta}(x)\end{aligned}$$

and the initial conditions

$$P_0^{\alpha,\beta}(x)=1,\quad P_1^{\alpha,\beta}(x)=[x(\alpha+\beta+2)+\alpha-\beta]/2. \tag{1.2}$$

When $\alpha>-1$ and $\beta>-1$, the Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the beta distribution $(1-x)^\alpha(1+x)^\beta\,dx$. When $\alpha+\beta\neq 0$, the Jacobi polynomials are well defined through (1.1), but when $\alpha+\beta=0$ one must be careful in defining the P_1. If we use (1.2), it is then clear that $P_1=x+\alpha$. On the other hand, if we let $\alpha+\beta=0$ in (1.1), then use (1.1) with the initial conditions $P_{-1}=0$ and $P_0=1$ and compute P_1 from the recursion (1.1), we will see that in addition to the option $P_1=x+\alpha$ we may also choose $P_1=x$. The former choice leads to the standard Jacobi polynomials, [**8, 16**], while the latter choice

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