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ON TRANSFORMATIONS AND ZEROS OF POLYNOMIALS

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ABSTRACT. We survey certain transformations of the set $\pi_n[x]$ of n-th degree polynomials into themselves. These transformations share the property that polynomials with all their zeros in a certain real interval are mapped to polynomials with all their zeros in another real interval. Rich sources of such "zero-mapping" transformations can be found in the Laguerre-Pólya-Schur theory of multiplier sequences. Others follow from the theory of biorthogonal polynomials, by identifying them with a mapping from the parameter space. This identification leads to two general techniques for the generation of such transformations. As a consequence, we prove a result on the zeros of certain convolution orthogonal polynomials introduced by Al-Salam and Ismail.

1. Introduction. Let $\mathcal{T}$ be a transformation of the set of the n-th degree polynomials, forthwith denoted by $\pi_n[x]$, into itself. In general, even if it is known that all the zeros of u reside in a certain real interval, little can be said about the zeros' location of $\mathcal{T}u$. However, there exist many transformations that exhibit regularity in their "mapping" of zeros. Possibly the simplest nontrivial example is $\mathcal{T}u = u'$, which retains the property that all the zeros of u reside in a real interval (for a complex-plane version of this statement, the Gauss-Lucas theorem, cf. [**16**]). The themes of this paper are three general techniques to produce transformations that display interesting "zero-mapping" properties. The first is classical and its major elements can be traced back to the work of Laguerre, a century ago. Nonetheless, it deserves being better

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