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SHARP INEQUALITIES FOR THE PADÉ APPROXIMANT ERRORS IN THE STIELTJES CASE

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1. Introduction. For the first time, ten years ago, the inequalities in question were quietly used by the authors to prove the existence of valleys in the c-table [**6**, Formulae (29) and (31)]. It was so natural for us to consider that everything about the Padé approximants to the Stieltjes functions coming from Stieltjes' work was well known. However, in the literature [**1, 2, 3, 9**] we have not been able to find this!

Having discovered the above accident we proved the valley property in another way (not published) and two conjectured inequalities became "open problems" [**7**]. Today we can give a complete proof of these inequalities.

2. Main result. Let f be a nonrational Stieltjes function and A_m/B_n the $[m/n]$ Padé approximant to f. We call the differences $f-[m/n]$ "Padé approximant errors."

Theorem. *Let $[m/n]$ be a Padé approximant to the nonrational Stieltjes function f:*

$$f(z)=\int_0^{1/R}\frac{d\mu(t)}{1-tz},\quad d\mu\ge 0,\ R>0. \tag{1}$$

The following inequalities occur:

$\forall n\ge \max(0,-k)$ and $\forall x\in\,]0,R[$,

$$0<f(x)-[n+k+1/n](x)<\frac{x}{R}\{f(x)-[n+k/n](x)\},\quad k\ge -2, \tag{2}$$

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