ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 1, Winter 1991

ASYMPTOTICS FOR THE ZEROS OF THE PARTIAL SUMS OF e^z. I.

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Dedicated to Wolfgang J. Thron on the occasion of his seventieth birthday, August 17, 1988.

ABSTRACT. We continue the work of Szegö and others on describing the convergence of the zeros, $\{z_{k,n}\}_{k=1}^n$, of the normalized partial sum $s_n(nz)$ of e^z where $s_n(z) := \sum_{j=0}^n z^j/j!$, to the Szegö curve D_∞, where

$$D_\infty := \{z \in \mathbf{C} : |ze^{1-z}| = 1 \text{ and } |z| \le 1\}.$$

It turns out that the convergence rate of these zeros to D_∞ is exactly $O(1/\sqrt{n})$, as $n \to \infty$, whereas this convergence rate improves to $O((\log n)/n)$, as $n \to \infty$, on compact subsets of $\Delta\backslash\{1\}$, where $\Delta := \{z \in \mathbf{C} : |z| \le 1\}$. We further show that there are new curves, D_n, now depending on n, for which the zeros of $s_n(nz)$ are $O(1/n^2)$ in distance from the curve D_n, on any compact subset of $\Delta\backslash\{1\}$.

Included also are a number of figures which illustrate these results graphically.

1. Introduction. With $s_n(z) := \sum_{j=0}^n z^j/j!$, $n \ge 1$, denoting the familiar partial sum of the exponential function e^z, we investigate here the location of the zeros of the *normalized* partial sums, $s_n(nz)$, and the rate at which these zeros tend to the Szegö curve D_∞, defined by

$$D_\infty := \{z \in \mathbf{C} : |ze^{1-z}| = 1 \text{ and } |z| \le 1\}. \tag{1.1}$$

By way of review, the well-known Eneström-Kakeya Theorem (cf. Marden [**6**, p. 137, Exercise 2]) asserts that, for any polynomial $p_n(z) =$

* Research supported in part by the Air Force Office of Scientific Research and by the National Science Foundation.
Received by the editors on October 13, 1988, and in revised form on January 6, 1989.