ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 2, Winter 1991

INFINITESIMALLY GENERATED SUBSEMIGROUPS OF MOTION GROUPS

J. HILGERT

0. Introduction. Recent developments in nonlinear control theory (cf. [**2, 3**] etc.) and also in analysis (cf. [**19, 14, 15, 16, 17, 18**]) indicate that there is an increasing demand for a systematic Lie theory of semigroups. Whereas the groundworks of a local Lie theory begin to emerge (cf. [**12, 4, 5, 8**]), there is not much on the record on a global theory (cf. [**12, 6, 9**]). We will briefly outline the basic definitions and the principal difficulties.

Let G be a connected Lie group and S be a subsemigroup of G. In order to simplify matters we assume that the group generated by S in G algebraically is all of G. Then we can associate with S a tangent object $\underline{L}\,(S)$ by setting $\underline{L}\,(S) = \{x \in \underline{L}\,(G) : x = \lim_{n\to\infty} nx_n, \exp x_n \in S, n \in \mathbf{N}\}$, where $\underline{L}\,(G)$ is the Lie algebra of G and $\exp : \underline{L}\,(G) \to G$ is the exponential function. It turns out (cf. [**12**]) that $\underline{L}\,(S)$ is a wedge, i.e., that it is a closed convex set, which is also closed under addition and multiplication by positive scalars. Moreover it satisfies

$$e^{adx}\underline{L}\,(S) = \underline{L}\,(S) \quad \text{for all } x \in \underline{L}\,(S) \cap -\underline{L}\,(S), \tag{0.1}$$

where $adx(y) = [x, y]$ with the bracket in $\underline{L}\,(G)$. We call a wedge satisfying (0.1) a *Lie wedge* and $\underline{L}\,(S)$ the *tangent wedge* of S.

It has been shown in [**8**] that, for any Lie wedge W, there exists a local semigroup S_w with $\underline{L}\,(S_w) = W$, i.e., there is a neighborhood $\mathcal{U}$ of the identity in G containing S_w such that $S_wS_w \cap \mathcal{U} \subset S_w$ and $W = \{x \in \underline{L}\,(G) : x = \lim_{n\to\infty} nx_n \exp x_n \in S_w, n \in \mathbf{N}\}$. On the other hand the examples (cf. [**8**]) show that by no means is every Lie wedge in $\underline{L}\,(G)$ the tangent wedge of a (global) subsemigroup S of G. Thus the principal question is: For which Lie wedges W in $\underline{L}\,(G)$ do there exist subsemigroups S of G such that $\underline{L}\,(S) = W$?

It is one basic idea of Lie theory that the tangent object should provide as much information as possible on the object under consideration.

Received by the editors on March 18, 1986 and in revised form on September 2, 1987.