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THE DIAGONAL ENTRIES OF A HILBERT SPACE OPERATOR

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1. Introduction. Let T be a (bounded linear) operator acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. For each orthonormal basis (ONB) $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$, T admits a unique matrix representation of the form

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdot & \cdot & \cdot & t_{1n} & \cdot & \cdot \\ t_{21} & t_{22} & \cdot & \cdot & \cdot & t_{2n} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & & \\ \cdot & \cdot & & \cdot & & \cdot & & \\ \cdot & \cdot & & & \cdot & \cdot & & \\ t_{n1} & t_{n2} & \cdot & \cdot & \cdot & t_{nn} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \\ \cdot \\ \cdot \end{matrix}.$$

Let $\operatorname{diag}(T) = \{t_{11}, t_{22}, \dots, t_{nn}, \dots\}$ denote the *diagonal sequence* of T with respect to this basis. The diagonal entry t_{nn} is equal to $\langle Te_n, e_n\rangle$, and therefore it belongs to the *numerical range* of T,

$$W(t) = \{\langle Tx, x\rangle : x \in \mathcal{S}_1\},$$

where $\mathcal{S}_1 = \{x \in \mathcal{H} : ||x|| = 1\}$; moreover, if t is a limit point of the sequence $\{t_{nn}\}_{n=1}^{\infty}$, then t belongs to the *essential numerical range* of T,

$$W_e(T) = \cap\{W(T+K)^- : K \text{ is a compact operator}\}.$$

(See [**3,6**] for properties of $W(T)$ and $W_e(T)$. For instance, the well-known Toeplitz-Hausdorff theorem guarantees that $W(T)$ and $W_e(T)$ are convex sets.)

For which (necessarily bounded) sequences $\{a_n\}_{n=1}^{\infty}$ is it possible to find an ONB $\{e_n\}_{n=1}^{\infty}$ such that $\operatorname{diag}(T) = \{a_n\}_{n=1}^{\infty}$ with respect to this basis?

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