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VECTOR-VALUED LOCAL MINIMIZERS OF NONCONVEX VARIATIONAL PROBLEMS

PETER STERNBERG

In recent work with R.V. Kohn [**7**], a new and general method for obtaining local minimizers of variational problems was established. This technique uses the notion of Γ-convergence of functionals, first introduced by De Giorgi [**1**] in the 1960's, and yields existence of local minimizers to a Γ-convergent sequence of problems, provided, roughly speaking, that the limit problem possesses a local minimizer which is isolated. In this paper, I apply the method to establish existence of vector-valued local minimizers $u_\varepsilon : \Omega \to \mathbf{R}^2$ of the problem

$$(1) \qquad \inf_{u \in H^1(\Omega)} \int_\Omega W(u) + \varepsilon^2 |\nabla u|^2 \, dx,$$

for certain open, bounded sets $\Omega \subset \mathbf{R}^n$ and ε sufficiently small. Here $|\nabla u|^2 = |\nabla u_1|^2 + |\nabla u_2|^2$, $\partial\Omega$ is taken to be Lipschitz-continuous, and W is a nonnegative "double-well" potential vanishing at two points $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^2$.

In particular, such a minimizer will be a nonconstant solution of the Euler-Lagrange equation (system):

$$(2) \qquad 2\varepsilon^2 \Delta u = \nabla_u W(u) \quad \text{in } \Omega,$$

with the "natural" Neumann condition

$$\partial_n u = 0 \quad \text{on} \quad \partial\Omega.$$

Variational problems of form (1) arise in the so-called gradient theory of phase transitions [**5, 6**], as well as in studies of pattern selection [**8**]. The form of nonconstant local minimizers of (1) was first conjectured in [**8**].

A full definition of Γ-convergence is given below in (6) and (7), but the essential idea in this setting is to obtain the first term in an asymptotic

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