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GRADIENT THEORY OF PHASE TRANSITIONS WITH GENERAL SINGULAR PERTURBATIONS*

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1. Introduction. In this article we shall discuss some recent results concerning the gradient theory of phase transitions in a Van der Waals fluid. In particular, we shall be interested in singular perturbations of the free energy of a more general form than $\varepsilon^2|\nabla u|^2$, which has been studied in detail in [**4, 10** and **14**].

In the Van der Waals theory of phase transitions, the equilibrium states of the system are given by the minimizers of the total free energy $I(u) = \int_\Omega W(u)\,dx$, where $\Omega \subseteq \mathbf{R}^n$ is bounded and open, and $u : \Omega \to \mathbf{R}$ is the density of the fluid. The function $W : (0,\infty) \to \mathbf{R}$ is the free-energy per unit volume. We shall assume that W satisfies the following conditions

$W : (0,\infty) \to [0,\infty)$ is C^3,

$W(\tau) \to \infty$ as $\tau \to 0, \infty$,

$W(\tau) = 0$ if and only if $\tau = a, b$, where $0 < a < b < \infty$,

$W''(a) > 0$, $W''(b) > 0$.

Hence, W is a nonconvex function with two minima of equal heights at $u = a, b$. (For the case where the free-energy W has two local minima of different heights, we consider the integrand $W(\tau) - (\alpha\tau + \beta)$, for some $\alpha, \beta \in \mathbf{R}$.) We refer to a and b as the phases of the fluid.

We shall look for minimizers of I in the class of functions satisfying the constraint

$$\int_\Omega u(x)\,dx = M; \tag{1}$$

that is, the total mass of the fluid is specified. If $M \in (a|\Omega|, b|\Omega|)$, where $|\Omega|$ is the measure of Ω, then the minimizers of I subject to (1)

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