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ON ITERATED TORSION PRODUCTS OF ABELIAN p-GROUPS

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ABSTRACT. The question of when $\operatorname{Tor}(A_1, \dots, A_n)$ is a dsc for abelian groups $A_1, \dots, A_n$ is discussed. The proofs involve inductions on both the number and cardinality of the groups. When $A_1, \dots, A_n$ have countable length and cardinality $\aleph_n$, necessary and sufficient conditions are given using invariants from set theory.

0. Introduction. In this paper the term "group" will be used to mean an abelian p-group for some fixed prime p (except for a momentary consideration of cotorsion groups in Theorem 12). In [**12**] Nunke investigates the question of when $\operatorname{Tor}(A, B)$ is a direct sum of countable groups (dsc). He arrives at an answer (cf ., our Theorem 1) in the case where A and B have different lengths. The case where the lengths are the same was left unresolved. In that paper Nunke also looks at iterated torsion products with an eye toward constructing $\aleph_n$-*cyclic* groups (groups whose subgroups of cardinality strictly less than $\aleph_n$ are direct sums of cyclics). A special case of Nunke's question is considered by Hill [**5**]. He arrives at separate necessary and sufficient conditions for $\operatorname{Tor}(A, B)$ to be a direct sum of cyclic groups which essentially induct on the cardinality of the groups involved. These conditions are generalized in Keef [**7**] which considers when $\operatorname{Tor}(A, B)$ is a dsc whose length is a limit ordinal.

The purpose of the present paper is to generalize the above results in two ways. First, instead of simply considering when $\operatorname{Tor}(A, B)$ is a dsc, we look at the question of when $\operatorname{Tor}(A_1, \dots, A_n)$ is a dsc. This allows us not only to induct on the cardinality of the various groups, but also on the number of groups involved. We also discuss the case where the length of the groups involved is an isolated countable ordinal. Perhaps the most interesting result of the present paper is a determination of when $\operatorname{Tor}(A_1, \dots, A_n)$ is a dsc of countable length and cardinality $\aleph_n$ (Theorem 10).

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