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RIEMANN INTEGRATION IN BANACH SPACES

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In this paper we will consider the Riemann integration of functions mapping a closed interval into a Banach space. This problem was first studied by Graves [**4**]. As new results in the theory of Banach spaces appeared, various authors added to the theory of vector-valued Riemann integration. Most of the results in this paper are a compilation of the works of Graves [**4**], Alexiewicz and Orlicz [**1**], Rejouani [**9, 10**], Nemirovski, Ochan, and Rejouani [**7**], and da Rocha [**11**].

Many of the real-valued results concerning the Riemann integral remain valid in the vector case. However, in the vector case a Riemann integrable function need not be continuous almost everywhere. It is an interesting problem to determine which spaces have the property that every Riemann integrable function is continuous almost everywhere and an analysis of this problem will be one of the main focuses of this paper. We will also examine the relationship between the Riemann integral and other vector-valued integrals.

We begin with some terminology and notation. Throughout this paper X will denote a real Banach space and X^* its dual.

Definition 1. A partition of the interval $[a,b]$ is a finite set of points $\{t_i : 0 \le i \le N\}$ in $[a,b]$ that satisfy $a = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = b$. A tagged partition of $[a,b]$ is a partition $\{t_i : 0 \le i \le N\}$ of $[a,b]$ together with a set of points $\{s_i : 1 \le i \le N\}$ that satisfy $s_i \in [t_{i-1}, t_i]$ for each i. Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \le i \le N\}$ be a tagged partition of $[a,b]$. The points $\{t_i : 0 \le i \le N\}$ are called the points of the partition, the intervals $\{[t_{i-1}, t_i] : 1 \le i \le N\}$ are called the intervals of the partition, the points $\{s_i : 1 \le i \le N\}$ are called the tags of the partition, and the norm $|\mathcal{P}|$ of the partition is defined by $|\mathcal{P}| = \max\{t_i - t_{i-1} : 1 \le i \le N\}$. If $f : [a,b] \to X$, then $f(\mathcal{P})$ will denote the Riemann sum $\sum_{i=1}^N f(s_i)(t_i - t_{i-1})$. Finally, the (tagged) partition $\mathcal{P}_1$ is a refinement of the (tagged) partition $\mathcal{P}_2$ if the points

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