ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 3, Summer 1991

GENERALIZATIONS OF THE GLEASON–KAHANE–ŻELAZKO THEOREM

KRZYSZTOF JAROSZ

Let A be a commutative, complex Banach algebra with a unit and let M be a one codimensional subspace of A. A.M. Gleason [**5**] and, independently, J.P. Kahane and W. Żelazko [**8**] proved that

(∗) M is an ideal if and only if M consists only of noninvertible elements.

Equivalently, if each element f of M belongs to a proper ideal I_f, which may depend on f, then M is actually an ideal. There is one-to-one correspondence between one codimensional subspaces (ideals) of any unital Banach algebra A and one dimensional subspaces of $A^{\#}$, the space of all linear functionals on A (linear-multiplicative functionals), hence the Gleason–Kahane–Żelazko theorem can be formulated:

$\binom{*}{*}$ Let $F \in A^{\#}$. Then F is multiplicative if and only if for any $f \in A$ we have $F(f) \in \sigma(f)$,

where $\sigma(f)$ denotes the spectrum of f.

The aim of this note is to give the history of various extensions and generalizations of the above result and to present some open problems.

In 1968 W. Żelazko [**15**] proved that the statement $\binom{*}{*}$ holds for any complex Banach algebra not necessarily unital and commutative. The proof is strictly algebraic, showing that $F \in A^*$ is multiplicative if and only if F restricted to any commutative subalgebra of A is multiplicative.

Statements (∗) and $\binom{*}{*}$ are equivalent for any complex, unital Banach algebras, not necessarily commutative, but they are not equivalent for nonunital Banach algebras. Here are two simple examples.

Example 1. Let S be a locally compact, not σ-compact, Hausdorff space. Put $A_0 = C_0(S)$, the algebra of all continuous functions defined

Received by the editors on May 27, 1988, and in revised form on July 22, 1988.