ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 21, Number 4, Fall 1991

EXTENSIONS OF MODULES CHARACTERIZED BY FINITE SEQUENCES OF LINEAR FUNCTIONALS

URI FIXMAN AND FRANK OKOH

ABSTRACT. Let S be an algebra over an algebraically closed field, K. If S is different from K, then it contains $K^2 = K \oplus K$ as a K-vector subspace, e.g., $S = K[\zeta]$, the polynomial ring in one variable over K. Then any S-module M gives rise to a pair of K-vector spaces $\mathbf{M} = (M, M)$ and a K-bilinear map from $K^2 \times M$ to M. This makes M a right module over the matrix ring, $R = \begin{bmatrix} K & K^2 \\ 0 & K \end{bmatrix}$. An R-module isomorphic to $\mathbf{M} = (M, M)$ where M is a $K[\zeta]$-module is said to be nonsingular; an R-module is torsion-free if it is isomorphic to a submodule of $\mathbf{M} = (M, M)$ where M is a torsion-free $K[\zeta]$-module. In this paper it is shown that extensions X of finite-dimensional torsion-free R-modules U by nonsingular R-modules are characterized by finite sequences of linear functionals. This provides an upper bound on the dimension of the vector space of extensions of U by V. Questions about such extensions become questions on the existence of linear functionals with appropriate properties. In particular, when $V = (K(\zeta), K(\zeta))$, where $K(\zeta)$ is the $K[\zeta]$-module of rational functions the setup provides a fertile source of indecomposable infinite-dimensional R-modules. We describe extensions, X, of U by V, with the property that the endomorphism ring of X is an integral domain. Moreover, X shares an infinite-dimensional indecomposable submodule with V.

Introduction. We fix a field K which we assume to be algebraically closed, and, unless otherwise stated, we let all vector spaces, linear and bilinear maps be over K. That K is algebraically closed is often dispensable in the paper, but it is convenient. For instance, the set $B = \{1/(\zeta - \theta)^n : \theta \in K, n = 1, 2, \dots\} \cup \{\zeta^n : n = 0, 1, 2, \dots\}$ is a K-basis for $K(\zeta)$. If the set of positive prime numbers is replaced by the set $\{1/(\zeta - \theta) : \theta \in K\}$, then one sees that a characterization of the $K[\zeta]$-submodules of the $K[\zeta]$-module $K(\zeta)$ is given in Section 85 of [**7**]. With this characterization as a point of departure, many attempts have been made to classify other torsion-free $K[\zeta]$-modules, see Section 93

Received by the editors on September 30, 1988.