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EXISTENCE AND MULTIPLICITY RESULTS FOR A CLASS OF ELLIPTIC PROBLEMS WITH CRITICAL SOBOLEV EXPONENTS

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0. Introduction. In this paper we consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + K(x)|u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{0.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^n$ ($n \geq 3$) or a compact manifold with boundary, $2^* = 2n/(n-2)$ is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \subset L^p(\Omega)$ and K is a smooth function on Ω.

When $K(x) = 1$ and Ω is a domain, some remarkable results have been obtained: Brézis and Nirenberg proved in [**5**] existence of a positive solution of (0.1), with $n \geq 4$, for all $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue for the negative Laplacian in Ω under Dirichlet boundary conditions; in [**6**] it was proved that (0.1), with $n \geq 4$, has a solution for any $\lambda > 0$; later, in [**7**], the existence and multiplicity problem for (0.1) with λ near an eigenvalue λ_j was studied; their main result was that (0.1) has at least m_j pairs of solutions for $\lambda \in (\bar{\lambda}_j, \lambda_j)$, where m_j is the multiplicity of λ_j and the constant $\bar{\lambda}_j$ can be estimated.

Problem (0.1) has a deep root in Riemannian geometry and physics. If one deforms a metric conformally in a closed manifold $(\mathcal{M}^n, g)$ of dimension $n \geq 3$ by a positive function $u : \mathcal{M} \to \mathbf{R}$, then u satisfies the equation

$$\begin{cases} \frac{4(n-1)}{n-2}\Delta u + Ru + Ku^{(n+2)/(n-2)} = 0 & \text{on } \mathcal{M} \\ u > 0 & \text{on } \mathcal{M}, \end{cases} \tag{0.2}$$

where Δ and R are, respectively, the Laplacian and the scalar curvature with respect to the metric g. The function K represents the scalar curvature of the new metric $u^{4/(n-2)}g$. An outstanding geometric

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