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ON MAPS WITH DENSE ORBITS AND THE DEFINITION OF CHAOS

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1. Introduction and definitions. Though the concept of chaos in dynamics goes back to at least Poincaré, it has only been during the flood of activity of the past decade that precise definitions of chaotic behavior have come forth [**2,3,8,18**]. The object of this article is to examine the relationship between the axioms for the most popular definition of chaos in discrete systems. The focus will be on a definitive analysis in the case of one-dimensional manifolds.

Let M be a metric space and let the distance between two points $x, y \in M$ be denoted $|x-y|$. A discrete dynamical system at its simplest is the set of iterates of a map $Q : M \to M$, i.e., $\{Q^0, Q, Q^2, Q^3, \dots\}$, where Q^0 is the identity function and Q^n denotes Q composed with itself n times.

The *orbit* of a point $x \in M$ is the set $\{x, Q(x), Q^2(x), Q^3(x), \dots\}$ and will often be written $\{x_n\}_{n=0}^{\infty}$ or $\{x_n\}$ when there is no ambiguity about which map Q is being used. A point x is said to be *periodic* if $Q^n(x) = x$ for some positive n. The minimum such n is called the *period* of x.

Frequently, the map Q depends on further parameters σ in an index set Σ, and one studies the behavior of the orbits obtained from Q_σ as σ is varied. The definition of chaos that we will use can be motivated by the following standard example. Let M be the closed interval $[0, 1]$. For every number $\sigma \in [0, 4]$, define

$$Q_\sigma(x) = \sigma x(1-x) \qquad x \in [0, 1].$$

When $\sigma \in [0, 1]$ the orbit of any point converges to the fixed point 0; hence, the set $\{0\}$ is called an attractor. For $\sigma \in (1, 3]$ the orbit of $x \neq 0, 1$ converges to $(\sigma - 1)/\sigma$. As σ increases to 4, Q_σ undergoes bifurcations and many periodic orbits emerge and higher periods occur. For $\sigma = 4$, there are no longer any attractors. In fact, for any point x there are points arbitrarily close to x whose orbits drift far away from

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