ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

SPECTRAL ALGEBRAS

THEODORE W. PALMER

ABSTRACT. Spectral algebras are a class of abstract complex algebras which share many of the good properties of Banach algebras. In the commutative case they are precisely the class of abstract algebras having a full Gelfand theory. Any irreducible representation of a spectral algebra is strictly dense. Spectral algebras are defined and characterized in terms of spectral pseudo-norms and spectral subalgebras. Spectral algebras, spectral subalgebras and spectral pseudo-norms are shown to occur frequently in analysis.

It is also shown that when the spectral radius is finite valued, if it is either subadditive or submultiplicative, then it has both properties and that this occurs exactly for algebras which are spectral algebras and commutative modulo their Jacobson radicals.

The paper is written in an expository style.

1. Introduction. The spectrum is undoubtedly the most important concept in the theory of (linear, associative) algebras. This article is an exploration of those abstract (i.e., not necessarily normed) complex algebras in which the spectrum behaves as it does in Banach algebras. Examples which outline the boundaries of this theory are also given. The article is written in an expository fashion with minimal prerequisites. Proofs are complete and self contained except for some well-known arguments for which specific references are given.

We shall use $\mathbf{C}, \mathbf{R}, \mathbf{R}_+$ and $\mathbf{N}$ to denote the complex numbers, real numbers, nonnegative real numbers and natural numbers, respectively. In this article the word *algebra* will mean a ring, $\mathfrak{A}$, which is also a complex linear space under the same addition and satisfies $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for all $a, b \in \mathfrak{A}$ and $\lambda \in \mathbf{C}$. In the introduction, for ease of exposition, we will also assume that $\mathfrak{A}$ contains a multiplicative identity, 1, (in this case $\mathfrak{A}$ is said to be *unital*), but this assumption will not be made later. For any element, $a \in \mathfrak{A}$, we define its *spectrum* and *spectral radius* by

$$\mathrm{Sp}(a) = \{\lambda \in \mathbf{C} : \lambda 1 - a \text{ has no inverse in } \mathfrak{A}\},$$

Received by the editors on February 24, 1986, and in revised form on October 3, 1988.