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ON THE ZEROS OF POLYNOMIALS AND SOME RELATED FUNCTIONS

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ABSTRACT. We consider the zeros of a polynomial $P(z)$ together with those of $F(z) \equiv (z-a)P'(z) - \nu P(z)$ where a and ν are arbitrary complex constants, and we extend some results obtained by Obrechkoff and Weisner on the relations between these sets of zeros. These results are applied to the zeros of certain quasi-trigonometric polynomials.

1. Introduction. Let $P(z)$ be a polynomial of n-th degree with zeros at $z_1, z_2, \ldots, z_n$. For arbitrary ν and a set

$$F(z) = (z-a)P'(z) - \nu P(z). \tag{1.1}$$

We observe that if $\nu = 0$ then $F(z) = (z-a)P'(z)$, and if $\nu = n$ then $F(z)$ is the negative of the derivative of $P(z)$ with respect to the point a, see [**3**; Vol. 2, pp. 61-63]. Since much is known about the zeros of $F(z)$ in these two special cases, we may assume henceforth that $\nu \neq 0$ and $\nu \neq n$.

Theorem A. *In* (1.1) *set* $\nu = n/2$. *If all the zeros of* $P(z)$ *lie inside* (*on, outside*) *a circle* $|z-a| = r$, *then all the zeros of* $F(z)$ *lie inside* (*on, outside*) *the same circle.*

This Theorem was proved by Obrechkoff [**2**] and later independently by Weisner [**4**]. In fact, both [**2**] and [**4**] prove far more general theorems which contain Theorem A as a special case.

In [**1**] Theorem A was extended to include arbitrary ν to obtain

Theorem B. *Let* $P(z)$ *be an* n*-th degree polynomial, and let* $F(z)$ *be defined by* (1.1). *If* $\operatorname{Re}\nu \geq n/2$, *let* G *be the region* $|z-a| \geq r$. *If*

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