ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

WEIGHTED SPLINES AS OPTIMAL INTERPOLANTS

K. ŠALKAUSKAS AND L. BOS

ABSTRACT. We consider interpolation by C^1 cubic splines which minimize a weighted semi-norm. The weight function is piecewise constant but on a subdivision potentially different from that determined by the interpolation knots.

Introduction. Suppose that we are given a set of data (x_i, f_i), $1 \leq i \leq N$, with $x_1 < x_2 < \cdots < x_N$. As is well known [**1**], among all functions f which interpolate the data (i.e., $f(x_i) = f_i$, $1 \leq i \leq N$) and which have an absolutely continuous first derivative and square integrable second derivative, the one for which $\int_{x_1}^{x_N} (f^{(2)}(x))^2\, dx$ is a minimum, is the natural cubic interpolating spline. This function is twice continuously differentiable on $[x_1, x_N]$ and is such that its restrictions to each of the subintervals, $[x_i, x_{i+1}]$, is a cubic polynomial. The adjective "natural" indicates that it may be extended by straight lines to a C^2 function, on all of $\mathbf{R}$, whose second derivative is in $L_2(\mathbf{R})$. This condition is easily seen to be equivalent to having zero second derivatives at the end points, x_1 and x_N. The use of the functional $\int_{x_1}^{x_N} (f^{(2)}(x))^2\, dx$ is motivated by the fact that it is a linearization of the bending energy of a thin elastic rod of uniform stiffness. Although cubic splines have found widespread application, there are data sets for which natural splines are not appropriate. Figure 1 below illustrates one such example.

Because of this, the first author introduced in [**4**] the weighted cubic spline, minimizing instead the weighted functional (or semi-norm)

$$|v|^2 := \int_{x_1}^{x_N} w(x)(v^{(2)}(x))^2\, dx$$

in the hope of being able to choose a weighting for which the resulting interpolant is not as unexpectedly oscillatory. To motivate our choice

This work was supported by NSERC Canada grants A8521 and A8389, respectively.
Received by the editors on February 14, 1989 and in revised form on July 4, 1989.