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THE SWAP CONJECTURE

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Introduction. We are interested in studying relationships among the various generating sets for a finitely generated group. All groups considered in this paper are assumed to be finitely generated.

Definition. Let $\Gamma_n(G) = \{(g_1, \dots, g_n) |$ the set $\{g_1, \dots, g_n\}$ generates $G\}$ $r(G) =$ rank of $G = \min\{n | \Gamma_n(G) \neq \varnothing\}$.

The generating sets γ_1 and γ_2 are *Nielsen equivalent*, written $\gamma_1 \sim_N \gamma_2$, if there is a sequence of Nielsen transformations, without deletions or insertions, leading from γ_1 to γ_2. The generating sets γ_1 and γ_2 are *swap equivalent* if there is a sequence of elementary swaps leading from γ_1 to γ_2, where an elementary swap changes one element of $\Gamma_n(G)$ to another by changing a single entry.

It is easily checked that Nielsen equivalence implies swap equivalence but not conversely and examples abound of groups with many, even infinitely many, Nielsen classes. We are unaware of any group with more than one swap class, motivating the conjecture of the title.

The swap conjecture. Any two finite generating sets for G of the same cardinality are swap equivalent.

In this paper, we relate these notions to two well-studied invariants of a generating set, namely the relation module and the relation space group (our terminology), and verify the conjecture for certain classes of groups. In Section 1 we give topological proofs of several known properties of relation modules and relation space groups.

1. Definitions and basic properties.

Definition. For $\gamma = (g_1, \dots, g_n) \in \Gamma_n(G)$, the associated epimorphism ε_γ from the free group $F[x_1, \dots, x_n]$ to G is defined by $\varepsilon_\gamma(x_i) = g_i$. Then $N(\gamma) = \ker(\varepsilon_\gamma)$, $i_\gamma : N(\gamma) \hookrightarrow F(\gamma) = F[x_1, \dots, x_n]$,

$$\overline{N}(\gamma) = N(\gamma)/[N(\gamma), N(\gamma)] \quad \text{and} \quad \overline{F}(\gamma) = F(\gamma)/[N(\gamma), N(\gamma)].$$

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