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PROJECTIVE SUMMANDS OF NOT SINGULAR MODULES

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1. Introduction. Let R be an associative ring with identity. In [**3**], Harada defined the term non-cosmall as follows: For a left R-module M over an associative ring R, M is non-cosmall if and only if there is an epimorphism of a free left R-module onto M such that the kernel is not essential. It turns out that this is equivalent to saying that the module is not singular, and we will adopt that term in this paper. Oshiro [**6**] and Harada [**3**] considered the rings R for which every left not singular module has a projective direct summand and called this condition $(*)^*$. A module M is called an extending module (or is said to have the extending property) if every nonzero submodule of M is essential in a direct summand of M. Oshiro [**6**] gives a complete description of the rings which satisfy $(*)^*$ in the case that the ring has acc on left annihilators. He calls these rings co-H-rings. His theorem states that a ring R is a left co-H-ring if and only if every projective left R-module is an extending module if and only if every left R-module is a direct sum of a projective module and a singular module, if and only if every essential extension of a projective R-module is projective. We are concerned with rings which satisfy the more general condition that every finitely generated not singular module has a nonzero projective direct summand, and we will denote this condition by $(F)^*$. If the ring is left nonsingular and has finite left uniform dimension, we will show that the ring R satisfies $(F)^*$ if and only if R is an FGSP ring in the sense of Goodearl [**1**], i.e., a ring R is an FGSP ring if the singular submodule is a direct summand of every f.g. module. We will obtain results similar to those of Oshira and Harada for rings with finite uniform dimension. We also show that the bounded rings which satisfy $(F)^*$, which are semi-perfect with nil radical, and have a unique minimal projective module which cogenerates all the f.g. projective modules are FQF-3 rings. We also show that if R is a semi-perfect ring with $\cap J^n = 0$ which is FQF-3, and each of its homomorphic images is an FQF-3 ring, then each of the finitely generated modules over R

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