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DISSIPATIVE DECOMPOSITION OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. A general decomposition theorem that allows one to express an arbitrary differential polynomial as a sum of conservative, dissipative and higher order dissipative pieces is proved. The decomposition generalizes the dissipative decomposition of ordinary differential equations, but is no longer unique. The proof relies on the properties of certain generalizations of the standard symmetric polynomials known as multi-symmetric polynomials. The nonuniqueness of the decomposition is a consequence of the syzygies among the power sum multi-symmetric polynomials.

1. Introduction. In a previous paper, [**14**], we proved that any polynomial ordinary differential equation in one independent and one dependent variable can be decomposed into a conservative part, a dissipative part and higher order dissipative pieces. Subject to certain homogeneity requirements, the decomposition is unique; in particular, it determines a unique conservative component of such an equation. The goal of this paper is to investigate to what extent the conservative/dissipative decomposition of nonlinear ordinary differential equations generalizes to partial differential equations. We will prove that an analogous decomposition always exists for polynomial partial differential equations in one dependent variable, but there is no longer a corresponding uniqueness result. The present proof of the decomposition theorem relies on a transform developed by Gel'fand and Dikii [**4**], and the second author, [**15**], which, in analogy with the classical Fourier transform, reduces problems in differential algebra to problems in commutative algebra. In our case, the problem transforms to a result in the theory of "multi-symmetric polynomials," which are certain multi-variable generalizations of the standard symmetric polynomials studied by, among others, Junker [**6, 7, 8**], and MacMahon [**9**] almost a century ago. The proof of the general dissipative decomposition then rests on some basic formulas relating the multi-symmetric analogs of

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