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METRIC SPACES AND MULTIPLICATION OF BOREL SETS

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ABSTRACT. Let us say that the Borel sets of spaces X and Y multiply if each Borel set in the product space $X \times Y$ is a member of the product σ-algebra generated by Borel rectangles. We show that the Borel sets of a space X and a metric space Y multiply if and only if the Borel sets of X and D multiply, where D is a discrete space having the same weight as Y.

1. Introduction. The Borel sets $\mathcal{B}(X)$ of a topological space X are the smallest σ-algebra containing the open sets of X. If X and Y are topological spaces, then $\mathcal{B}(X) \times \mathcal{B}(Y)$ denotes the smallest σ-algebra on $X \times Y$ containing sets of the form $E \times F$, where E and F are Borel sets of X and Y, respectively. Always $\mathcal{B}(X) \times \mathcal{B}(Y) \subset \mathcal{B}(X \times Y)$; if $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$, we say that the *Borel sets of* X *and* Y *multiply*. Notice that the Borel sets of X and Y multiply if and only if each open set in $X \times Y$ is a member of $\mathcal{B}(X) \times \mathcal{B}(Y)$.

Lemma 1.1. *If* Z *is a subset of* Y *and* $\mathcal{B}(X \times Y) = \mathcal{B}(X) \times \mathcal{B}(Y)$, *then* $\mathcal{B}(X \times Z) = \mathcal{B}(X) \times \mathcal{B}(z)$.

Proof. (Compare with [**6**, Theorem 7.1].) It is easily seen that since $\mathcal{B}(X) \times \mathcal{B}(Z)$ is a σ-algebra of subsets of $X \times Z$, the family

$$\mathcal{M} = \{M \subset X \times Y : (X \times Z) \cap M \in \mathcal{B}(X) \times \mathcal{B}(Z)\}$$

forms a σ-algebra of subsets of $X \times Y$. If $U \times V$ is an open rectangle in $X \times Y$, then $(X \times Z) \cap (U \times V) = U \times (Z \cap V)$ is an open rectangle in $X \times Z$, so $U \times V \in \mathcal{M}$ and, consequently, $\mathcal{M}$ contains $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$.

Now suppose W^* is open in $X \times Z$. There exists an open subset W of $X \times Y$ such that $W^* = (X \times Z) \cap W$. Since $W \in \mathcal{M}$, we have that $W^* \in \mathcal{B}(X) \times \mathcal{B}(Z)$. This implies that $\mathcal{B}(X \times Z) = \mathcal{B}(X) \times \mathcal{B}(Z)$. □

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