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SOME SERIES REPRESENTATIONS OF $\zeta(2n+1)$

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1. Introduction. For $\operatorname{Re}(s) > 1$ the Riemann zeta function $\zeta(s)$ is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It is well known that $\zeta(s)$ can be continued analytically to the whole complex plane except for a simple pole at $s = 1$ with residue 1. Moreover, $\zeta(0) = -1/2$.

In [**2**] Boo Rim Choe gives an elementary proof of the classical result

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{1.1}$$

by making use of the power series expansion of $\arcsin x$. In [**4**] Ewell modifies Boo Rim Choe's method to give a new series representation of $\zeta(3)$, namely,

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}. \tag{1.2}$$

Then in [**5**] Ewell further modifies the method of Boo Rim Choe to obtain the following representation of $\zeta(r)$ (valid for an integer $r > 2$):

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{m=0}^{\infty} (-1)^m A_{2m}(r-2)\pi^{2m}/(2m+2)!. \tag{1.3}$$

The coefficients $A_{2m}(r)$ are given by

$$A_{2m}(r) = \sum \frac{\binom{2m}{2i_1, 2i_2, \ldots, 2i_r}}{(2i_1+1)(2(i_1+i_2)+1)\cdots(2(i_1+i_2+\cdots+i_{r-1})+1)} \cdot B_{2i_1}B_{2i_2}\cdots B_{2i_r},$$

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