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TRIANGLE CENTERS AS FUNCTIONS

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ABSTRACT. We consider a kind of problem that appears to be new to Euclidean geometry, since it depends on an understanding of a *point* as a *function* rather than a position in a two-dimensional plane. Certain special points we call *centers*, including the centroid, incenter, circumcenter, and orthocenter. For example, the centroid, as a function of the class of triangles with sidelengths in the ratio $a_1 : a_2 : a_3$, is given by the formula $1/a_1 : 1/a_2 : 1/a_3$. The kind of problem introduced here leads to functional equations whose solutions are centers.

1. Introduction. A triangle $\Delta A_1A_2A_3$ with respective sidelengths a_1, a_2, a_3 and angles $\alpha_1, \alpha_2, \alpha_3$ (as in Figure 1) is often studied by means of homogeneous coordinates, as introduced by Möbius [**6**]; for a historical account, see Boyer [**1**]. In many discussions of triangles, homogeneous *barycentric* coordinates are preferred, but here we shall use homogeneous *trilinear* coordinates instead. The main reason for this choice is that our results depend on a formula for the distance between two points, and this formula (4a) is much shorter in trilinears than in barycentrics. Another reason is that a single reference (Carr [**2**]) gives many useful formulas in terms of trilinears, whereas no comparable reference seems to exist for barycentric formulas. Typical representations in trilinears, written as $x_1 : x_2 : x_3$ and defined in Section 2, are the following:

$$\begin{aligned}
\text{centroid} \quad & x_1 : x_2 : x_3 = 1/a_1 : 1/a_2 : 1/a_3 \\
\text{circumcenter} \quad & x_1 : x_2 : x_3 = a_1(a_2^2 + a_3^2 - a_1^2) : a_2(a_3^2 + a_1^2 - a_2^2) : \\
& a_3(a_1^2 + a_2^2 - a_3^2) = \cos\alpha_1 : \cos\alpha_2 : \cos\alpha_3 \\
\text{circumcircle} \quad & a_1/x_1 + a_2/x_2 + a_3/x_3 = 0 \\
\text{Euler line} \quad & x_1 \sin 2\alpha_1 \sin(\alpha_2 - \alpha_3) + x_2 \sin 2\alpha_2 \sin(\alpha_3 - \alpha_1) \\
& \quad + x_3 \sin 2\alpha_3 \sin(\alpha_1 - \alpha_2) = 0
\end{aligned}$$

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