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VALUATED VECTOR SPACES, KUREPA'S HYPOTHESIS AND ABELIAN p-GROUPS

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Dedicated to Professor Djuro Kurepa on the occasion of his 86th birthday.

ABSTRACT. The key result states that, for a regular cardinal $\varkappa$, the $\varkappa$-Kurepa hypothesis (the existence of a tree of height $\varkappa$ with levels of cardinality $< \varkappa$ and at least $\varkappa^+$ branches) is equivalent to the existence of a valuated vector space V of cardinality $\varkappa$ with the following properties: (a) its $\varkappa$-topology is Hausdorff; (b) for every $i < \varkappa$, $|V/V(i)| < \varkappa$; and (c) the completion $\tilde{V}$ of V in the $\varkappa$-topology has cardinality greater than $\varkappa$. Another equivalence to the $\varkappa$-Kurepa hypothesis is obtained by replacing (b) by the following condition (b$'$): For every $i < \varkappa$ and every subspace $W \leq V/V(i)$, with $|W| < \varkappa$, its closure $\overline{W}$, in the i-topology, also satisfies $|\overline{W}| < \varkappa$.

This is used to prove in a short and elegant way some results previously established by P. Keef; namely, Kurepa's hypothesis is equivalent to the existence of a C_{ω_1}-group G of length ω_1 and cardinality at least $\aleph_2$ with a p^{ω_1}-pure subgroup A of cardinality $\aleph_1$ whose closure in the ω_1-topology of G has cardinality at least $\aleph_2$. This is also equivalent to the existence of a C_{ω_1}-group of length ω_1 and balanced projective dimension 2.

Let V be a valuated vector space over a field F with valuation $v : V \to \text{Ord} \cup \{\infty\}$, i.e., a function satisfying $v(a) = \infty$ if and only if $a = 0$, $v(ta) = v(a)$ for all scalars $t \neq 0$, and $v(a+b) \geq \min\{v(a), v(b)\}$. Then by $V(\alpha)$ we mean the subspace $V(\alpha) = \{x \in V : v(x) \geq \alpha\}$. If λ is a limit ordinal, then by the λ-*topology* on V we mean the linear topology having as a base for the neighborhoods of 0 the set $\{V(\alpha) : \alpha < \lambda\}$. All the topologies in this paper will be of this kind. It is easy to see that if $a, b \in V$ with $v(a) \neq v(b)$ then $v(a + b) = \min\{v(a), v(b)\}$.

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