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ON TWO EXTREMAL PROBLEMS RELATED TO UNIVALENT FUNCTIONS

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ABSTRACT. For an integrable $\Lambda : [0,1] \to \mathbf{R}$, nonnegative on $(0,1)$, and $f \in \mathcal{S}$, the class of normalized univalent functions in the unit disk $\mathbf{D}$, we are interested in the functional

$$L_\Lambda(f) := \inf_{z\in\mathbf{D}} \int_0^1 \Lambda(t)\left(\operatorname{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt,$$

and, in particular, in $L_\Lambda(\mathcal{S}) := \inf_{f\in\mathcal{S}} L_\Lambda(f)$. Note that $L_\Lambda(\mathcal{S}) \le 0$ for every Λ. We show that $L_\Lambda(f) \ge 0$ for f close-to-convex and a set of functions Λ containing $\Lambda_c(t) := (1-t^c)/c$, $c \in (-1,2]$. This result turns out to be instrumental for our solution of the following problem: find the best (least) bound β_c so that for each $g \in \mathcal{H}(\mathbf{D})$ with $g(0)=0$, $g'(0)=1$, $\operatorname{Re}[e^{i\alpha}(g'(z)-\beta)] > 0$ in $\mathbf{D}$ with $\beta \ge \beta_c$ the function

$$(c+1)\int_0^1 t^{c-1} g(tz)\,dt, \qquad z \in \mathbf{D},$$

is starlike univalent in $\mathbf{D}$. Weaker bounds for β_c have been obtained by a number of authors (cf. Ali [**1**], Nunokawa [**6**]). We are using the duality principle for Hadamard products to obtain our results.

1. Introduction and statement of the results. Let $\mathcal{S}$ denote the set of univalent functions f in the unit disk $\mathbf{D}$, normalized by $f(0)=0$, $f'(0)=1$. The Koebe distortion theorem then states that, for $f \in \mathcal{S}$,

$$\frac{1}{(1+|z|)^2} \le \left|\frac{f(z)}{z}\right| \le \frac{1}{(1-|z|)^2}, \qquad |z|<1.$$

Generally, however, we do not have

$$\frac{1}{(1+|z|)^2} \le \operatorname{Re}\frac{f(z)}{z}, \tag{1}$$

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