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ON SOME INEQUALITIES INVOLVING $(n!)^{1/n}$

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When investigating a conjecture on an upper bound for permanents of $(0,1)$-matrices, H. Minc and L. Sathre [**2**] (see also [**1**]) obtained several inequalities involving $f(n) = (n!)^{1/n}$–the geometric mean of the first n positive integers. One of their results is

Theorem A. *If* $n \geq 1$ *is an integer, then*

$$1 < \frac{f(n+1)}{f(n)} < 1 + \frac{1}{n}. \tag{1}$$

Another one, "probably the most interesting ..., and certainly the hardest to prove" [**2**, p. 41] is

Theorem B. *If* $n \geq 2$ *is an integer, then*

$$1 < n\frac{f(n+1)}{f(n)} - (n-1)\frac{f(n)}{f(n-1)}. \tag{2}$$

The aim of this note is to establish sharpenings of inequalities (1) and (2). We present a lower bound for the difference on the right-hand side of (2) which is greater than 1. Furthermore, we give an answer to the question: What is the largest real number α and the smallest real number β such that

$$1 + \frac{\alpha}{n+1} \leq \frac{f(n+1)}{f(n)} < 1 + \frac{\beta}{n+1}$$

is valid for all integers $n \geq 1$?

First we provide a monotonicity theorem.

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